Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 1, pp. 54–59, 2004

MATHEMATICAL MODEL FOR THE NUMERICAL SOLUTION OF NONSTATIONARY PROBLEMS IN SOLID MECHANICS BY A MODIFIED GODUNOV METHOD

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UDC 519.683.2

A mathematical model of substance behavior under developed elastoplastic strains is worked out for solving one-dimensional problems of solid mechanics. The model is based on the fundamental laws of conservation of mass, momentum, and total energy, Wilkins model, kinetic model of substance destruction, and modified Godunov method for the numerical solution of problems in mathematical physics. A hybrid difference scheme is constructed, which approximates acoustics equations with constant coefficients in smooth flows for the case of plane symmetry with the second order in time and space.

Key words: *dynamics, elastoplastic strain, destruction, numerical solution.*

Introduction. A mathematical model of the behavior of ideally plastic solid isotropic materials under conditions of developed elastoplastic strains is considered for solving one-dimensional nonstationary problems of solid mechanics by the modified Godunov method [1, 2]. It is assumed that the material is isothermal, Poisson's ratio is constant, and there are no body forces. The governing equations of the model are the laws of conservation l of mass, momentum, and total energy. This system of equations is supplemented by constitutive relations for the description of elastoplastic properties of substances in the Prandtl–Reuss form under the von Mises flow rule [3] and by equations of material damage and destruction.

This paper involves consideration of two destruction models, i.e., the kinetic model of metal destruction [4] developed at the Institute of Machine Science of the Ural Division of the Russian Academy of Sciences and based on the concept of gradual accumulation of microdamages in the metal in the course of strain evolution and the model of instantaneous spalling based on the concept of material destruction on reaching the critical stress.

To improve the accuracy of numerical solutions, regulated approximational antidiffusion is introduced for correction of numerical fluxes [5] obtained by the Godunov method. The modified difference scheme on smooth solutions approximates acoustics equations with constant coefficients in the case of plane symmetry with the second order in time and space.

The difference equations of motion are derived with allowance for consistent approximation [6], which, in combination with the difference equation for the total energy, ensures approximation of the internal energy equation. Because of this, there is no effect of partial transfer of kinetic energy to internal energy.

A hybrid difference scheme is constructed for solving one-dimensional problems of elastoplasticity in the Lagrangian variables by the modified Godunov method.

Test calculations were performed and the numerical results were compared with experimental data. The results presented show that the modified Godunov method with the use of the model of substance destruction can be applied to solve problems of shock and high-velocity impact.

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Mathematical Model. The governing equations for solving problems with a single space derivative are written in the form of the laws of conservation of mass, momentum, and total energy:

$$\frac{\partial w}{\partial t} + \frac{\partial (FQ)}{\partial q} = f_2, \qquad \frac{\partial r(q,t)}{\partial t} = u(q,t). \tag{1}$$

In these laws,

$$w = \begin{bmatrix} V \\ u \\ \varepsilon \end{bmatrix}; \qquad F = \begin{bmatrix} -u \\ -\sigma_1 \\ -\sigma_1 u \end{bmatrix}; \qquad f_2 = \begin{bmatrix} 0 \\ -\sigma_2\, \partial Q/\partial q \\ 0 \end{bmatrix};$$

r is the Euler coordinate, q is the mass Lagrangian variable, t is the time, $V = 1/\rho$ is the specific volume, ρ is the density, u is the component of the velocity vector, $\varepsilon = E + 0.5u^2$ is the total energy, E is the specific internal energy of the substance, σ_1 and σ_2 are the components of the tensor of principal stresses, and $Q(r) = r^\nu$ ($\nu = 0$, 1, and 2 for plain, cylindrical, and spherical symmetry, respectively).

System (1) is supplemented by the equations of state of the substance $p = p(\rho, E)$ and constitutive relations for the stress-tensor deviator, which are written in the form of the Prandtl–Reuss equations [3]

$$\frac{dS_j}{dt} - 2Ge_j + \lambda S_j = 0 \qquad (j = 1, 2, 3),$$

where S_j are the components of the stress-tensor deviator, G is the shear modulus, and λ is a scalar dissipative function.

The model of an ideal elastoplastic Mises body with a constant yield point is described by the function

$$\lambda = \begin{cases} 0 & \text{if } I_1 < 0 \text{ or } I_1 = 0 \text{ and } D < 0, \\ 3GD/\sigma_s^2 & \text{if } I_1 > 0 \text{ or } I_1 = 0 \text{ and } D > 0. \end{cases}$$

Here $I_1 = S_1^2 + S_2^2 + S_3^2$, $D = S_1e_1 + S_2e_2 + S_3e_3$, e_1, e_2, and e_3 are the components of the deviator of the strain-rate tensor, and σ_s is the yield point under uniaxial tension.

The kinetic model of metal destruction is based on the concept of gradual accumulation of microdamages in the metal in the course of strain evolution [4]. The condition of plastic deformation of the material without destruction is written in the form of the differential equation

$$\frac{d\Psi}{dt} = \frac{H}{\Lambda_p},$$

where $H = 2\sqrt{|e_1e_2 + e_3e_1 + e_3e_2|}$ is the intensity of the shear strain rate, $\Lambda_p(k_1, k_2, H)$ is the material plasticity, $0 \leqslant \Psi \leqslant 1$ is the damage, and k_1 and k_2 are the dimensionless parameters of the strain state. The time of material destruction corresponds to the time when Ψ reaches unity.

The simplest model of instantaneous spalling is based on the concept of material destruction on reaching the critical stress $\sigma \geqslant \sigma_{\text{cr}}$.

Difference Scheme. At the initial time $t = t_0$, we construct a grid with the coordinates (nodes) $q_{j+1/2}$ and $q_{j-1/2}$, with the centers at the points $q_j = 0.5(q_{j+1/2} + q_{j-1/2})$, and with the space steps $h_j = q_{j+1/2} - q_{j-1/2}$ [$h_j = \rho_j(R_{j+1/2}^{\nu+1} - R_{j-1/2}^{\nu+1})/(\nu+1)$ is the mass Lagrangian variable; $R_{j+1/2}$ and $R_{j-1/2}$ are the values of the Euler variable in the grid nodes]. The quantities $w^j = [(V, u, E, \rho, \Psi)^j]^{\text{t}}$ with an integer superscript j correspond to the points q_j (cell centers) and are calculated at the times $t_n + \tau$ from the values of $w_j = [(V, u, E, \rho, \Psi)_j]^{\text{t}}$ with the subscript j specified at the times t_n. The quantities $F_{j\pm1/2}^* = [(U, \Sigma_1, \Sigma_1 U)_{j\pm1/2}^*]^{\text{t}}$ with a half-integer subscript are intermediate and correspond to the points $q_{j+1/2}$ and $q_{j-1/2}$. Here Σ_1^* is the value of the first component σ_1 of the tensor of principal stresses and U^* is the velocity; $[\ldots]^{\text{t}}$ indicates transposition of the row-vector.

After writing Eqs. (1) in the form of integral conservation laws, we integrate them over a certain domain S: $q \in [q_{j+1/2}, q_{j-1/2}]$ and $t \in [t_n, t_n + \tau]$. Applying the Gauss–Ostrogradskii theorem to the integrals over the domain, we obtain a difference scheme, which is written in the vector form as

$$w^j = w_j - \tau(F_{j+1/2}^* - F_{j-1/2}^*)/h + \tau f_2^0 V_g^0 \tag{2}$$

($f_2^0 = [0, -\sigma_2^0, 0]^{\text{t}}$). The quantity σ_2^0 will be defined below. The quantities $F_{j\pm1/2}^*$ are intermediate and are calculated at the boundary of two neighboring cells. The method of choosing the intermediate values determines the specific difference scheme. Thus, in the Godunov method, these values are called "high" and are calculated from the

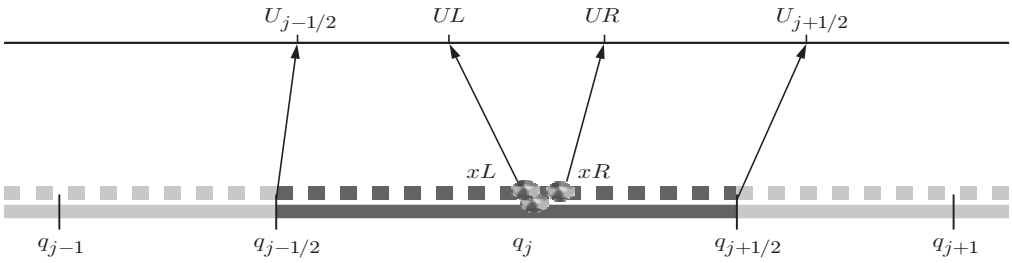

Fig. 1. Crack model in the cell.

solution of the problem of decay of an arbitrary discontinuity arising on the boundary of two neighboring cells. The components of the vector $F^*_{j\pm1/2}$, which is denoted by $F_{j\pm1/2}$ in the Godunov method, are calculated by the formulas [7]

$$U_{j+1/2} = \frac{a_{j+1}u_j + a_j u_{j+1}}{a_{j+1}+a_j} + \frac{\sigma_{1j+1}-\sigma_{1j}}{a_{j+1}+a_j}, \qquad \Sigma_{1\,j+1/2} = a_{j+1}a_j\,\frac{u_{j+1}-u_j}{a_{j+1}+a_j} + \frac{a_{j+1}\sigma_{1j+1}+a_j\sigma_{1j}}{a_{j+1}+a_j}$$

($a_j = \rho_j c_j$ is the mass velocity of sound and c_j is the longitudinal velocity of sound).

For $u_j < 0$, we have $V_{j+1/2} = V_j + (\sigma_{1j} - \Sigma_{1j+1/2})/a_j^2$ and $S_{1\,j+1/2} = S_{1j} - 4b_j^2(1/V_{j+1/2} - 1/V_j)/3$. For $u_j > 0$, we obtain $V_{j+1/2} = V_{j+1} + (\sigma_{1j+1} - \Sigma_{1j+1/2})/a_{j+1}^2$ and $S_{1\,j+1/2} = S_{1\,j+1} - 4b_{j+1}^2(1/V_{j+1/2} - 1/V_{j+1})/3$. Here, b_j is the transverse velocity of sound.

The system of governing difference equations (2) is supplemented by the equations

$$((S_i)^j - (S_i)_j)/\tau - 2G(e_i)^0_j + \lambda_j(S_i)^0_j = 0 \qquad (i=1,2), \qquad (S_3)_j = -(S_1+S_2)_j.$$

The kinetic equation allowing for destruction is approximated by the difference equation

$$\Psi^j = \Psi_j + \tau H(F^*_{j+1/2}, F^*_{j-1/2})/\Lambda_{pj}.$$

The value of $\Lambda_{pj} = \Lambda_p = \Lambda_p(k_j)$ can be obtained from the table of experimental data in [4].

Antidiffusion. One method of improving accuracy of numerical solutions obtained by the Godunov method is to introduce antidiffusion for correcting numerical fluxes. To improve the accuracy of solutions obtained from scheme (2), we use the approach suggested by Moiseev in [5]. According to [5], the components of the vector $F^*_{j\pm1/2}$ are calculated by the formulas

$$F^*_{j+1/2} = F_{j+1/2} - \mu(\tau_{j+1/2} - 0.5\tau)\,\frac{\partial f}{\partial t}. \tag{3}$$

The second addend in (3) has the meaning of antidiffusion with respect to $F_{j\pm1/2}$ [5]. The parameter $0 \leqslant \mu \leqslant 1$ is the delimiter of antidiffusion, and its value is chosen depending on the flow parameters, thus, enabling one to obtain quasi-monotonic profiles of the quantities being calculated.

By virtue of the equations obtained by linearization of Eqs. (1) for the plane case, the time derivative $\partial f/\partial t$ is replaced by the space derivative. The space derivative can be approximated by a combination of difference derivatives calculated at the time t_n (at the lower level) and at the time $t_n + \tau_{j+1/2}$ from the values of $F_{j\pm1/2}$. If the derivative is approximated over the values from the lower level, we obtain a scheme with normal dispersion. If the derivative is approximated using the values of $F_{j\pm1/2}$ found from the solution of the arbitrary discontinuity decay, we arrive at a new scheme with anomalous dispersion. In the first case, the solution oscillations in the vicinity of a strong discontinuity occur behind the discontinuity front; in the second case, they arise ahead of this front.

For $\mu = 0$ in (3), we have $F^*_{j+1/2} = F_{j+1/2}$ and the difference scheme (2) approximates the original differential equations on smooth solutions for the case of plane symmetry with the first-order in time and space, it does not require any additional artificial viscosity, and allows obtaining monotonic solutions for a wide class of problems [5, 8].

For $\mu = 1$ in (3), scheme (2) approximates acoustics equations with constant coefficients on smooth solutions for the case of plane symmetry with the second-order in time and space. Taking into account the above-mentioned properties, we construct a hybrid difference scheme where the values of the parameter μ are chosen close to unity for calculating smooth flows and close to zero on strong discontinuities.

Consistent Approximation. As is known, the original Godunov scheme with the equation of motion written in a divergent form involves the effect of partial transfer of kinetic energy to internal energy. It is shown

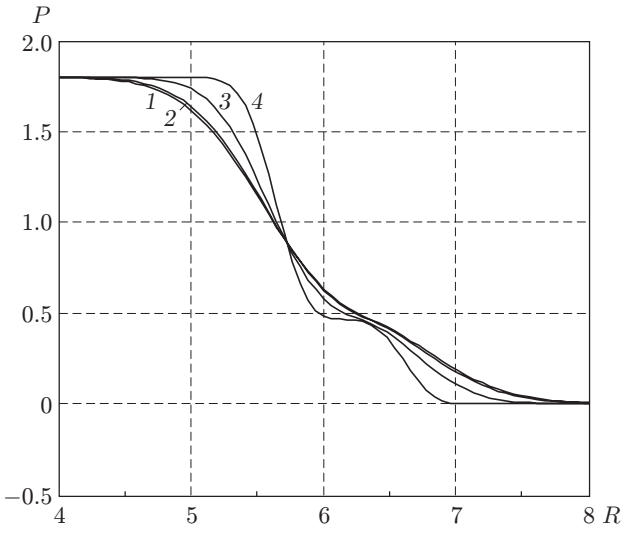

Fig. 2

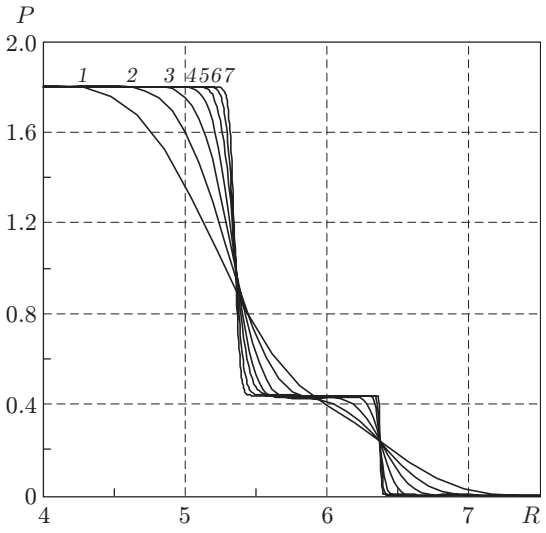

Fig. 3

Fig. 2. Pressure distribution over the space coordinate: $\mu = 0$ (1), 0.1 (2), 0.5 (3), and 0.9 (4).

Fig. 3. Pressure distribution versus the number of division points N ($\mu = 0.9$ and $C = 0.9$): $N = 62$ (1), 125 (2), 250 (3), 500 (4), 1000 (5), 2000 (6), and 4000 (7).

in [6] that this effect can be eliminated if the calculation of pressure in the right side of the equation is consistent with the calculation of "high" values of pressure at the cell edges. In this case, the pressure is chosen so that the difference equations approximating equations of motion in divergent and nondivergent forms should be equivalent. In accordance with this approach, the quantities in the right side are calculated from the equations

$$\sigma^0_{1j} = -p_0 + S^0_1 = 0.5(\Sigma^*_{1\,j+1/2} + \Sigma^*_{1\,j-1/2}) = -0.5(P^*_{j+1/2} + P^*_{j-1/2}) + 0.5(S^*_{1\,j+1/2} + S^*_{1\,j-1/2}).$$

The second component of the deviator S^0_{2j} is found from the condition $I_1(S) = S^0_1 + S^0_2 + S^0_3 = 0$ and is equal to

$$S^0_{2j} = -0.5S^0_{1j} = -0.25(S^*_{1\,j+1/2} + S^*_{1\,j-1/2}).$$

Such a choice of values makes it possible to eliminate partial transfer of kinetic energy to internal energy and provides approximation of the internal energy equation

$$\frac{\partial E}{\partial t} - Q\sigma_1 \frac{\partial u}{\partial q} = \sigma_1 u \frac{\partial Q}{\partial q} - u(S_1 - S_2) \frac{\partial Q}{\partial q}$$

in the difference scheme (2).

Destruction Process. The process of high-velocity impact of solids is accompanied by versatile physical phenomena depending on the geometry of interacting bodies, their strength characteristics, impact velocity, and other factors. In particular, materials are mechanically destroyed under the action of tensile stresses. To take destruction into account, numerical methods employ various destruction criteria. The destruction processes of the substance in a cell can be modeled in different ways [9]. We use two models in the present paper. In the first model, the original material in the cell is substituted by a powder substance, which experiences no shear and tensile resistance. The pressure and tangential stresses are assumed to be equal to zero. In the second model, crack formation is simulated in a cell where substance destruction occurred: the cell is divided into two subcells whose left and right boundaries $q_j = \{xL, xR\}$ correspond to crack edges (Fig. 1).

If $xR > xL$, the crack exists. If $xR < xL$, the crack is closed. The boundary condition of the free-surface type is imposed on these boundaries, and then the subcells are calculated in a usual manner.

Calculation Results of Model Problems. We give some results of numerical calculations.

Problem 1. *The Numerical Calculation Results are Compared with the Analytical Solution.* For barotropic processes of medium motion, we consider the equation of state of the form [10] $P = A(\rho/\rho_0 - 1)$. The following material constants are used: $A = 75$ GPa, $\rho_0 = 2.7$ g/cm^3, $\nu = 0.333$, $\sigma_s = 0.3$ GPa, and $G = 26$. The boundary conditions are

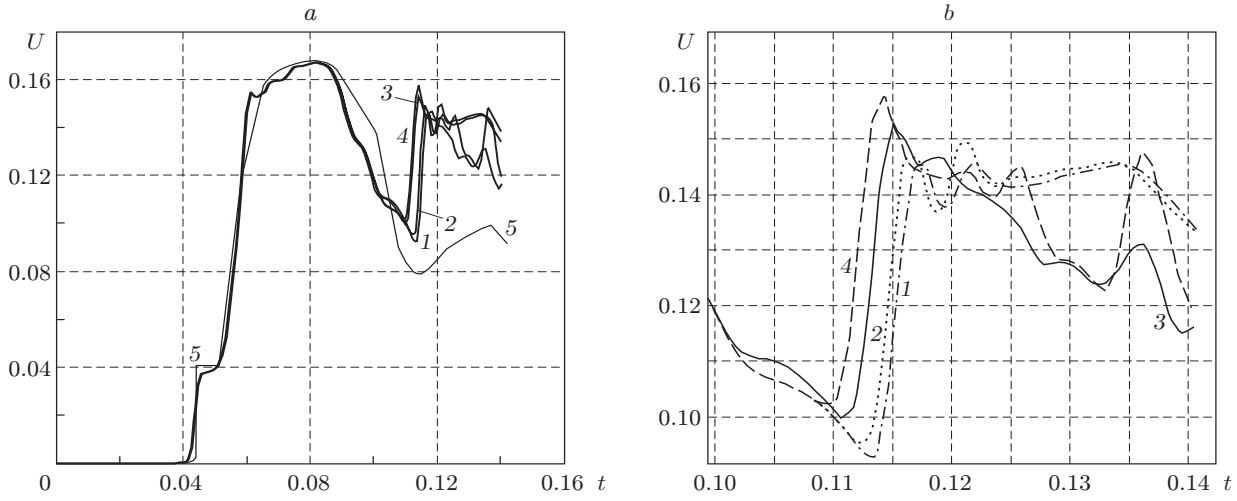

Fig. 4. Calculated velocity of the uranium free surface versus time: 1) model of instantaneous spalling with cracks; 2) model of instantaneous spalling with an undamaged cell; 3) kinetic model with cracks; 4) kinetic model with an undamaged cell; 5) experimental data of [11].

$$p(t)\Big|_{R=0} = 2 \text{ GPa}, \qquad p(t)\Big|_{R=12} = 0.$$

The calculation results are shown in Figs. 2 and 3.

Problem 2. *Modeling of Spalling Destruction of Uranium under Conditions of High-Velocity Impact.* A system consisting of an impactor and a target is set in the calculation. The impactor and target material is uranium. The impactor velocity is $w_{\text{im}} = 0.1750$ km/sec. In the initial state, the impactor density is $\rho = \text{const}$; $P = 0$. The target is at rest, and its initial density is $\rho = \text{const}$; $P = 0$. The target thickness $\Delta_{\text{t}} = 1.513$ mm is two times greater than the impactor thickness $\Delta_{\text{im}} = 0.768$ mm. Spalling in uranium is to be numerically modeled. The equations of state are chosen in the Mie–Grüneisen form. The material constants are $\rho_0 = 19.04$ g/cm^3, $\nu = 0.205$, $\sigma_s = 0.9$ GPa, and $G = 88.0$. The antidiffusion parameter is $\mu = 0.9$, and the Courant number is $C = 0.9$. The number of division points is $N_{\text{im}} = 50$ for the impactor and $N_{\text{t}} = 100$ for the target. The experimental data for this problem are taken from [11]. The calculation results are plotted in Fig. 4. Fig. 4b shows a scaled-up fragment of Fig. 4a. It is seen that the free-surface velocity depends on the destruction model used (Fig. 4b). At the initial stage of motion, the numerical results are in good agreement with the experimental data, but a certain difference is observed after passing the first minimum (Fig. 4a).

It follows from the presented results of solving test problems that the numerical solutions obtained by the above-described mathematical model agree with analytical solutions and experimental data. The analysis of the results also shows that the kinetic model of substance destruction can be used to solve problems of shock and high-velocity impact.

REFERENCES

1. S. K. Godunov , "Difference method for numerical calculation of discontinuous solutions of hydrodynamic equations," *Mat. Sb.*, **47**, No. 3, 271–306 (1959).
2. S. K. Godunov, A. V. Zabrodin, M. Ya. Ivanov, et al., *Numerical Methods for Solving Multidimensional Problems in Gas Dynamics* [in Russian], Nauka, Moscow (1976).
3. M. L. Wilkins, "Calculation of elastoplastic flows," in: B. Alder, S. Fernbach, and M. Retenberg (eds.), *Methods of Computational Physics*, Vol. 3, Academic Press, New York (1964).
4. V. L. Kolmogorov, *Mechanics of Metal Forming* [in Russian], Izd. Ural. Gos. Tekh. Univ., Ekaterinburg (2001).
5. N. Ya. Moiseev, "Antidiffusion for the modified Godunov method," in: *Computational Mechanics and Modern Applied Program Systems*, Abstracts of Xth Int. Anniversary Conf. (Pereyaslavl'-Zalesskii, June 7–12, 1999), Moscow (1999).

6. N. Ya. Moiseev, "Consistent approximation in difference schemes of S. K. Godunov type for solving one-dimensional problems of gas dynamics," *Zh. Vychisl. Mat. Mat. Fiz.*, **36**, 149 (1996).
7. V. N. Demidov and A. I. Korneyev, "Numerical method for calculating elastoplastic flows using moving difference grids," Tomsk University, Tomsk (1983). Deposited at VINITI, No. 2924 (1983).
8. M. Kh. Abuzyarov, V. G. Bazhenov, V. L. Kotov, et al., "Method of discontinuity decay in dynamics of elastoplastic media," *Zh. Vychisl. Mat. Mat. Fiz.*, **40**, No. 6, 940–953 (2000).
9. V. M. Fomin, A. I. Gulidov, G. A. Sapozhnikov, et al., *High-Velocity Interaction of Bodies* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (1999)
10. V. A. Batalov, V. A. Svidinskii, V. I. Selin, and V. N. Sofronov, "Software UP for solving one-dimensional gas-dynamic and elastoplastic problems of continuous media," *Vopr. Atom. Nauki Tekh., Ser. Metod. Progr. Chisl. Resh. Mod. Mat. Fiz.*, No. 1, 21–28 (1978).
11. S. Cochran and D. Banner, "Spall studies in uranium," *J. Appl. Phys.*, **48**, No. 7, 2729–2737 (1977).